Deferred-Acceptance Heuristic Auctions*

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Abstract

We study auctions in which allocations are decided by an iterative process of rejecting the least attractive remaining bids. These "deferred-acceptance heuristic auctions" have distinctive properties that make them attractive for applications in computationally challenging environments. Deferred acceptance "threshold" auctions are group strategy-proof, can be implemented using clock auctions, and are outcome-equivalent in our complete-information model to paid-as-bid auctions based on the same heuristic. Paid-as-bid auctions based on such heuristics are dominance solvable, and every non-bossy dominance-solvable paid-as-bid auction is a deferred-acceptance heuristic auction. None of these properties are shared by auctions based on optimization or greedy-acceptance heuristics.

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1 Introduction

Over the last twenty years, economic theory has contributed to tremendous progress in the development of practical auctions for buying or selling multiple differentiated items. The earliest applications were to sales of radio spectrum licenses, but many more have followed.[1] A major challenge on the frontier of research and development today is to design and build auctions for computationally challenging resource allocation problems— ones for which it may be impossible to compute exactly optimal allocations. The leading application in recent years has been to combinatorial auctions, but an even newer application motivates this study. It arises from the US government's effort to conduct what it calls an "incentive auction," to reallocate frequencies from television broadcasting to wireless broadband services.[2] The planned transaction involves sharing a portion of the proceeds of the sale of wireless broadcast licenses with current television broadcasters in order to provide them an "incentive" to relinquish their licenses, while reassigning the remaining over-the-air broadcasters to a smaller set of channels. The reassignment of broadcasters who do not sell is a source of the enormous complexity, because it must be done so that no two broadcast stations are assigned to channels in ways that create interference between them. There are more than 100,000 such constraints. Just determining whether a given

[1] The "simultaneous multiple round" design used in the first US spectrum auctions is described by FCC economist Evan Kwerel in the Foreword to Milgrom (2004). Additional discussions are found in McMillan (1994) and Milgrom (2000). The "combinatorial clock auction" design recently adopted in many European countries as well as Australia and Canada was proposed by Ausubel, Cramton and Milgrom (2006). A sealed bid auction allowing bidders to express substitution (the "product-mix" or "assignment" auction) was introduced by Milgrom (2009) and Klemperer (2010) and adopted by the Bank of England for securities auctions. Variations of these several designs have been adopted in markets for electricity, natural gas, water rights, diamonds, Internet domain names, gaming rights, among others.

[2] According to estimates by the Congressional Budget Office, the "forward" auction will raise about $40 billion in gross revenues, of which about $15 billion will be paid to broadcasters who sell their over-the-air broadcasting rights in the "reverse" auction.

set of broadcast stations can feasibly be assigned to channels is a computationally challenging problem, equivalent to the NP-hard "graph coloring problem" (see Aardal et al. (2007) for a survey of computational approaches to this problem). The problem of selecting a feasible set of stations to leave on-air so as to maximize their total value is even harder in practice, and cannot currently be solved exactly, even using weeks of computer time with state-of-the-art algorithms and hardware. If an effective auction for the spectrum reallocation problem can be designed and implemented, that might also open a path to devise auctions for other large, highly constrained resource allocation problems, such as flight scheduling for aircraft or joint scheduling for aircraft and commercial space flights.

Computationally challenging mechanism design problems have so far mostly been studied by computer scientists in the growing field of "Algorithmic Mechanism Design," which originated with the work of Nisan and Ronen (1999). The typical goal is to devise mechanisms that are strategically sound, generate good economic outcomes, and can be computed in reasonable time.[3] In designs like the Vickrey auction, in which prices are determined as the difference between two optimum values, and even small percentage errors of optimization can lead to very large pricing errors, destroying strategy-proofness. Mechanisms based directly on heuristics can sometimes ensure exact strategy-proofness, even without computing exactly optimal outcomes.

The leading approach to auctions based on heuristics is one pioneered by Lehmann, O'Callaghan and Shoham (2002) (hereafter LOS), who introduce strategy-proof mechanisms that employ a "greedy-acceptance" heuristic algorithms.[4] For the problem of selling a set of heterogeneous goods, a

[3]Economists have long been concerned about the computational properties of economic allocation mechanisms, starting at least with Hayek (1945). However, the economic literature has focused on formal modeling of communication costs (e.g. Hurwicz 1977, Mount and Reiter 1974, Nisan and Segal 2006, Segal 2007), which are trivial in the present setting of single-minded bidders, while the computational burden could be overwhelming.

[4]Variants of greedy-acceptance heuristic auctions have also been studied – see, e.g., Mu'alem and Nisan (2008), Babaioff and Blumrosen (2008), and references therein.

"greedy-acceptance" heuristic prioritizes the bids according to some "score," and iteratively accepts the highest-scoring bid that is still feasible. For their analysis, LOS introduce the important concept of a single-minded bidder – one who is interested in buying just one particular package of goods – and determine the unique payments to winning bidders that make truthful reporting a dominant strategy for all single-minded bidders.

In this paper, we introduce a class of auctions based on a similar but different class of heuristics and investigate their economic and game-theoretic properties. Like the greedy-acceptance heuristics, the alternative heuristics prioritize bids for consideration, but the processing begins with the "least attractive" bids for the auctioneer instead of the "most attractive" ones. Instead of greedily accepting the most attractive bids, the alternative heuristics greedily reject the least attractive bids and, when the algorithm terminates, the bids that were not rejected are finally accepted. To create a strategy-proof auction when bidders are single-minded, each winner's payment is set to its "threshold price," which is its least attractive bid that would have still won, given the bids of the others. We call the combination a "deferred-acceptance threshold auction" because it is a close cousin to the Gale-Shapley deferred-acceptance algorithm.[5] In our setting with monetary bids, deferred-acceptance threshold auctions are also closely related to clock auctions, which offer rejected bidders an opportunity to "improve" their price offers, and at the end accept all the standing bids that are not rejected. For numerous

[5] In the Gale-Shapley two-sided matching algorithm, the side that receives offers rejects all but the best offers at each round and only at the end accepts the offers that were not rejected. In its clock auction version, the bidders in a round of a deferred-acceptance threshold auction offers its most preferred potentially acceptable terms and then moves down its list to less preferred ones, with the auctioneer rejecting raising the bar of acceptability at each round.

Just as this paper contrasts the properties of deferred-acceptance and greedy-acceptance mechanisms, Abdulkadiroglu, Pathak, Roth and Sonmez (2005) contrast the Gale-Shapley mechanism with the greedy-acceptance-based "Boston mechanism." In the latter mechanism, in each round, schools accept their highest ranked student applicants until the school's capacity is reached or until the algorithm ends in a round with no new offers.

examples of deferred-acceptance heuristics and/or clock auctions, see Kelso and Crawford (1982), Moulin (1999), Ausubel (2004), Gul and Stacchetti (2000), Milgrom (2000), Hatfield and Milgrom (2005), de Vries et al. (2007), Juarez (2009), Mehta et al. (2009), Ensthaler and Giebe (2009, 2010) and Bikhchandani et al. (2011).

Despite the obvious similarity between the greedy-acceptance and deferred-acceptance algorithms – one greedily accepts attractive bids, rejecting the remainder, and the other greedily rejects unattractive bids, accepting the remainder – the classes of auctions based on these algorithms have some very different game-theoretic and economic properties that are important in practice. All deferred-acceptance threshold auctions are group strategy-proof and can be implemented using clock auctions, but the class of greedy-acceptance threshold auctions has neither property.[6] All paid-as-bid auctions based on "non-bossy" deferred-acceptance heuristics are dominance solvable, but no such property holds for the class of non-bossy greedy-acceptance heuristics.[7] Another difference emerges in an outcome-equivalence property: when the same non-bossy deferred-acceptance heuristic is used to select winners for either threshold auctions or paid-as-bid auctions, the complete-information undominated Nash equilibrium outcomes for the two mechanisms coincide, but the same is not generally true when winners are selected by a greedy-acceptance heuristic.

Yet another property of deferred-acceptance heuristics is that every monotonic

[6] There are some classes of special cases in which the greedy-acceptance and deferred-acceptance rules make identical choices, so the auctions based on them coincide. For example, if the rule is to take the k highest bids from a set of n (with $n > k$), that can be implemented by either a greedy-acceptance or a deferred-acceptance algorithm, and the greedy-acceptance auctions share the properties of the deferred-acceptance auctions in those cases.

[7] "Non-bossiness" is a common condition in mechanism design theory. An allocation rule is "non-bossy" if no change in a participant's report can alter another participant's winning/losing status without altering its own status as well. In deferred-acceptance auctions, this means making a losing bid worse does not change whether any other bid is winning.

allocation rule in which bids are price-theoretic substitutes, and that satisfies a "no-disposal" property, can be computed exactly using some such heuristic. This capability is not shared by greedy-acceptance heuristics. For the spectrum reallocation problem, while many interference constraints influence the assignments of stations to channels, the most expensive constraints appear to be the limits to the number of channels available in each metropolitan area. Those constraints tend to make stations substitutes, which suggests that some deferred-acceptance auction might perform well. To test that, the FCC conducted simulations with the actual constraints and internally constructed value data. Using a simple deferred-acceptance heuristic, it found that the value of stations taken off-air is only slightly higher that from a long run of their optimization algorithm. The difference was less than one percent.

In an important departure from previous studies, we consider an unusually wide class of heuristics in which the prioritization of bids is not fixed in advance but rather can be flexibly adjusted during the auction depending on the identities of the previously rejected bidders as well as their monetary bids. This flexibility is crucial for both the theoretical and practical parts of our analysis. For the theory, this wider class is needed to prove several of the previous results, including (i) the equivalence between the classes of deferred-acceptance heuristics and clock auctions, (ii) the computability of substitutable, "no-disposal" allocation rules by deferred-acceptance heuristics, and (iii) that the winning bids in any monotonic, non-bossy, dominance-solvable paid-as-bid auction are selected by some deferred-acceptance heuristic. For the practice, the wider class is necessary and useful for designing auctions that incorporate binding budget constraints (or binding revenue targets) for the auctioneer, as well as ones that incorporate "yardstick competition" among bidders to reduce costs (or increase revenues), as in Segal (2003).

For concreteness, we focus on procurement ("reverse") auctions, which correspond to the application to radio spectrum reallocation. The deferred-acceptance heuristic then begins with an excess-supply situation and proceeds iteratively, rejecting the highest-scoring bids (with scores increasing in the bid amounts) until excess supply is eliminated. Our results can also be adapted to apply to selling ("forward") auctions, which begin with excess-demand situations and iteratively reject the lowest-scoring bids until excess demand is eliminated. Combining both versions, our analysis applies to all the studies of clock auctions cited above, several of which focus on forward auctions.

The paper is organized as follows: Section 2 describes the general class of deferred-acceptance heuristics for processing bids and gives a number of examples. Section 3 defines clock auctions in which winners are paid their final clock prices, and shows that, when bidders are restricted to use only cutoff strategies, every clock auction is equivalent to some sealed-bid deferred-acceptance threshold auction, and vice versa. Section 4 shows that, in private-value environments, clock auctions are strategy-proof and (weakly) group strategy-proof: there is no individual or coalitional deviation from truthful bidding that makes all the deviators strictly better off. These strategy-proofness results hold regardless of what information is disclosed during the auction, from full disclosure in one extreme to no disclosure in the other. By the previous equivalence, it then follows that deferred-acceptance threshold auctions are group strategy-proof. Section 5 answers a question about the kinds of allocation rules that are exactly implementable using a deferred-acceptance procedure. It shows that any monotonic allocation rule that treats bids as substitutes and has a "no-disposal" property can be implemented by some deferred-acceptance heuristic.

Section 6 compares the performance of paid-as-bid and threshold auctions designs using under the assumption of complete information and assuming that both auctions use the same deferred-acceptance heuristic to select winning bids. This comparison provides one indicator of the possible cost of replacing the familiar paid-as-bid auction by a strategy-proof auction. It finds that the dominant strategy solution of the threshold auction is has the same allocation and prices as some Nash equilibrium outcome of the paid-as-bid auction. While a paid-as-bid auction will typically have other Nash equilibria as well, we show that for any non-bossy deferred-acceptance heuristic, the paid-as-bid auction is dominance-solvable for generic values of the bidders (that is, its payoffs are completely determined by iterated elimination of dominated strategies) and that this dominance solution coincides with the equilibrium described above. Moreover, under the same conditions, every Nash equilibrium in undominated strategies leads to the same outcome.[8] The distinctiveness of this dominance solvability property is highlighted by the following converse: any dominance-solvable paid-as-bid auction that selects winners using a monotonic, non-bossy allocation rule must implement the same outcome mapping as some deferred-acceptance heuristic.

In Section 7, we demonstrate by means of examples that the properties of group strategy-proofness and paid-as-bid outcome-equivalence are not guaranteed when winners are selected using either a greedy-acceptance heuristic or optimization. We conclude with a discussion of the several results that have been useful for evaluating design options for the FCC's "incentive auction."

2 Heuristic Sealed-Bid Auction

Let N be the set of bidders. In the auction, each bidder either "wins" (which means that his bid to supply a given good or set of goods is "accepted") or

[8]These outcome equivalence results are related to the findings of Bernheim and Whinston (1986) about the equivalence between paid-as-bid auctions and Vickrey auctions for the case in which the allocation is chosen to maximize the total bid and bidders are substitutes. One connection is that, for that case, the Vickrey auction is implementable by a deferred-acceptance heuristic. A difference is that Bernheim and Whinston (1986) select a particular Nash equilibrium by imposing a coalition-proofness refinement while we instead select an equilibrium using iterated deletion of dominated strategies.

"loses" (which means that his bid is rejected). We restrict attention to auctions in which winners receive payments but losers do not. The preferences of each bidder i depend on whether he wins or loses, and, when he wins, on the payment p_i. We assume that these preferences are strictly increasing in the payment, and there exists some payment v_i that makes him indifferent between winning and losing, which we call his "value". (For unmixed outcomes, such a preference can be expressed by a quasilinear utility p_i when the bidder wins and v_i when he loses.) The set of bidder i's possible values is $[0, \overline{v}_i]$ for each i.

A mechanism requests each bidder i to submit a bid $b_i \in B_i \subseteq \mathbb{R}$ and generates a set of $A \subseteq N$ of accepted bids for every bid profile $b \in B = \Pi_i B_i$. We restrict each bid space B_i to be a closed set such that $\sup B_i > \overline{v}_i$.[9] (Below, we often further restrict B_i to be finite.) Let $\alpha : B \to 2^N$ denote the winner determination rule or "allocation rule" of the mechanism: $\alpha(b) \subseteq N$ is the set of winners generated for bid profile $b \in B$.

A deferred-acceptance heuristic is a particular kind of mechanism described by a set of scoring functions, as follows. For each set $A \subseteq N$ of active bidders and each bidder $i \in A$, there is a *scoring function* $s_i^A : B_i \times B_{N \setminus A} \to \mathbb{R}_+^A$ that is nondecreasing in its first argument. The heuristic then operates as follows. Let $A_t \subseteq N$ denote the set of active bids in stage t. Initialize $A_1 = N$. For each $t \geq 1$, if $s_i^{A_t}\left(b_i, b_{N \setminus A_t}\right) = 0$ for all $i \in A_t$ then stop and output A_t, otherwise let $A_{t+1} = A_t \backslash \arg\max_{i \in A_t} s_i^{A_t}\left(b_i, b_{N \setminus A_t}\right)$ and continue. Intuitively, the heuristic process is one of iteratively deleting the least desirable (highest scoring) bids until only zero scores remain.

[9] The last restriction ensures that each bidder prefer to participate in the auction. Alternatively, we could restrict attention to auctions in which each bidder has a "non-participation bid" that loses against any profile of other bids. In this case, a bidder i with value $v_i \geq \sup B_i$ will submit a non-participation bid. The bidder's maximum bid that has the possibility of winning could then be interpreted as that bidder's "reserve price".

2.1 Payments and Strategy-proofness

A *payment rule* is a function $p : B \to \mathbb{R}^N$ specifying that agent i receives $p_i(b)$. A *sealed-bid auction* is a triple $\langle B, \alpha, p\rangle$ such that $p_i(b) = 0$ whenever $i \in N\backslash\alpha(b)$ (meaning that losing bidders are not paid).

Since we will often use finite bid sets, it is convenient to replace the usual notion of "truthful" bidding by a similar concept of "strategy-proofness" that applies even when some possible bidder values do not correspond to feasible bids. According to our definition, an auction is strategy-proof if it is always optimal for a bidder to round up its value to next lowest allowable bid. With this definition, if the sets of possible values and bids are both the same interval of real numbers, then strategy-proofness and truthfulness coincide.

Definition 1 *The sealed-bid auction $\langle B, \alpha, p\rangle$ is* strategy-proof *if for every bidder i, $v_i \in [0, \bar{v}_i]$, and $b_{-i} \in B_{-i}$, it is optimal for bidder i to bid $v_i^+ \equiv \min\{b_i \in B_i : b_i > v_i\}$.*

Definition 2 *The allocation rule α is* monotonic *if and only if $i \in \alpha(b_i, b_{-i})$ and $b_i' < b_i$ imply $i \in \alpha(b_i', b_{-i})$.*

With these definitions, a standard argument implies the following:

Lemma 3 *A sealed-bid auction $\langle B, \alpha, p\rangle$ is strategy-proof if and only if α is monotonic and payments satisfy the following formula for all $b \in B$, $i \in \alpha(b)$:*

$$p_i(b_{-i}) = \sup\{b_i' \in B_i : i \in \alpha(b_i', b_{-i})\}. \tag{1}$$

It is easy to see that any deferred-acceptance heuristic generates a monotonic allocation rule and that the corresponding threshold prices (1) to the winners can be computed as follows: Start with $p_i^0 = \sup B_i$ for all i, and then for each round $t \geq 1$, compute

$$p_i^t(b) = \min\left\{p_i^{t-1}, \sup\left\{b_i' \in B_i : s_i^{A_t}\left(b_i', b_{N\setminus A_t}\right) < s_j^{A_t}\left(b_j, b_{N\setminus A_t}\right) \text{ for } j \in A_t\setminus A_{t+1}\right\}\right\}$$

for every bidder $i \in A_{t+1}$. In the final round of the algorithm, for every winner $i \in A^T$, $p_i^T(b)$ is the winner's threshold price.[10]

Inspection of the formula shows a consequential property of the threshold prices for deferred-acceptance heuristics: holding fixed the final set of winners A_T, winning bidders' threshold prices depend on the losing bids $b_{N\setminus A_T}$ but *not* on the winning bids b_{A_T}. It follows that no winning bidder can affect another winner's threshold price except by changing to a losing bid.

2.2 Examples

Example 4 (Feasibility Constraint) *Let $F \subseteq 2^N$ denote the set of sets of bidders that could be feasibly accepted, and assume that $N \in F$, so that the procurement goal is achievable. To ensure that the heuristics maintains feasibility, we require that $s_i^A\left(b_i, b_{N\setminus A}\right) > 0$ only if $A\setminus\{i\} \in F$, and also that there are no ties, i.e., $s_i^A\left(b_i, b_{N\setminus A}\right) \neq s_j^A\left(b_j, b_{N\setminus A}\right)$ for all $i \neq j$, A, $b_i, b_j, b_{N\setminus A}$.*

We say that the heuristic has perfect feasibility checking *if $s_i^A\left(b_i, b_{N\setminus A}\right) > 0$ if and only if $A\setminus\{i\} \in F$ – i.e., it stops only when all active bids are infeasible to reject. In some settings, however, perfect feasibility checking may be too computationally challenging, and imperfect checking must be used instead. For example, in the FCC's spectrum-clearing problem, to check whether a given set A of bidders is in F requires checking whether there exists an assignment of the rejected bidders $N\setminus A$ to available channels that satisfies all interference constraints, and this is an NP-hard problem. When a feasibility checker has a limited time to run, it may generate three possible outputs: (i)*

[10] These round-by-round computations can be integrated with the heuristic in one single calculation step.

establish that $A \backslash \{i\} \in F$ by generating a feasible assignment of the rejected bidders to channels, (ii) establish that $A \backslash \{i\} \notin F$ by generating a proof that such a feasible assignment does not exist, and (iii) be timed out before generating either (i) or (ii). In case (i), we set $s_i^A\left(b_i, b_{N \backslash A}\right) > 0$, while in cases (ii) and (iii), we need to set $s_i^A\left(b_i, b_{N \backslash A}\right) = 0$, to guarantee that the heuristic yields a feasible assignment.

The next two examples show two reasons why it may be useful to condition the scoring functions on the rejected bids. The first reason is to incorporate a budget restriction that makes the auctioneer reject additional bids when the current overall costs are too high. The second reason is to create "yardstick competition" among bidders by inferring reasonable reserve prices for the active bidders from the rejected bids.

Example 5 (Budget or Payment Constraint) *Suppose that the total payment to the winners cannot exceed $R(A)$ when the set of accepted bids is A. For example, in the FCC case, payments to broadcasters are limited by the revenue obtained from selling the cleared spectrum in the forward auction, net of some expenditures required by statute or regulations. Since the FCC may be initially uncertain about how much spectrum it can clear subject to a net payment constraint, it might set a sequence of n possible procurement goals represented by feasible sets: $F_1 \subset ... \subset F_n$ with the corresponding forward auction revenues $R_1, ..., R_n$, so that the maximum forward auction revenue achieved by accepting set A of bids is $R(A) = \max\{R_k : 1 \leq k \leq n,\ A \in F_k\}$.) Then in every round t, the algorithm could look at the total threshold prices that would have to be paid to each of the active bidders in A_t if the algorithm were to stop in this round, and continue to the next round if this total exceeds $R(A_t)$. For a simple example, if scores are based on functions $\sigma_i(b_i) > 0$ (independent of the comparison set A and of others' bids), the threshold price that would have to be paid to a currently active bidder $i \in A$ if the heuristic* *were to stop right away can be calculated as:*

$$p_i\left(b_{N\setminus A}\right)=\sup\left\{b_i'\in B_i:\sigma_i\left(b_i'\right)<\max_{j\in N\setminus A}\sigma_j\left(b_j\right)\right\}.$$

To add a "stopping rule" that allows the auction to end only if the budget constraint is met, we let for each $i\in A$,

$$s_i^A\left(b_i,b_{N\setminus A}\right)=\begin{cases}0 & \text{if } \sum_{j\in A}p_j\left(b_{N\setminus A}\right)\le R\left(A\right),\\ \sigma_i\left(b_i\right) & \text{otherwise.}\end{cases}$$

This can be viewed as a "revenue-sharing"' problem, which is the mirror image of the "cost-sharing"' problem of Moulin (1999) and Mehta et al. (2009). Our formulation permits the auction to generate revenue in excess of $R(A)$ to be absorbed by the auctioneer, but it is possible to modify it to require revenue to be exactly $R(A)$. (This possibility will become clearer once we introduce clock auctions and show (Proposition 8) that for every clock auction there exists an equivalent deferred-acceptance threshold auction.)

Example 6 (Reference Pricing) *Suppose the auctioneer cares about the expected total profit, and that his gross profit for acquiring each bidder is π. Suppose that bidders' values are drawn i.i.d. from a distribution that is unknown to the auctioneer. In this symmetric setting we can focus without loss on symmetric auctions. If we consider symmetric deferred-acceptance heuristics that do not condition on the rejected bids, these heuristics accept all bids above some fixed reserve price p^* (e.g. setting $s_i^A(b_i)=\max\{v_i-p^*,0\}$), which yields an expected profit on each bidder i of $(\pi-p^*)\Pr\{v_i\le p^*\}$, where the probability is calculated based on the auctioneer's prior. Note, however, that these expected profits could be improved by conditioning the reserve price $p_A^*\left(b_{N\setminus A}\right)$ in each round on the rejected bids (i.e., letting $s_i^A\left(b_i,b_{N\setminus A}\right)=\max\left\{v_i-p_A^*\left(b_{N\setminus A}\right),0\right\}$). For example, the reserve price could be the optimal price for the beliefs about the distribution of values of the active bidders that are updated based on $b_{N\setminus A}$. (Note: the expected profit-maximizing threshold* *auction, described in Segal (2003), implements the allocation rule* $\alpha(b) = \{i \in N : b_i \leq p(b_{-i})\}$ *where* $p(b_{-i}) \in \arg\max_p (\pi - p) \Pr\{v_i \leq p|b_{-i}\}$*. If the family of possible distributions of bidders' values is ordered in the likelihood ratio ordering, then* $p(b_{-i})$ *is nondecreasing in* b_{-i}*, hence allocation rule* α *has the substitute property. If in addition the range of* α *has no disposal, then by Proposition 14 below* α *can be implemented as a heuristic auction.)*

3 Clock Auction

Informally, a ("descending") clock auction is a dynamic mechanism that proposes a declining sequence of price offers to each bidder, with each offer followed by a decision period in which any bidder whose price has been strictly reduced can decide to exit or continue. Bidders that have never exited are called "active"; others are called "inactive." Bidders who remain active when their prices are reduced are said to "accept" the lower price. When the auction ends, the active bidders become the winners and they are paid their last (lowest) accepted prices. Different clock auctions are distinguished by their pricing functions, which determine the sequence of prices to offer the several bidders.

To formalize the intuitive description, we restrict attention to heuristics with finite bid spaces and to finite clock auctions with discrete periods.[11] The active bidders in period t are denoted by $A_t \in 2^N$. A period-t history consists of the sets of active bidders in all periods up to period t: $A^t = (A_1, ..., A_t)$ such that $A_t \subseteq ... \subseteq A_1 = N$. Let H denote the set of all such histories. A descending clock auction is a price mapping $p : H \to \mathbb{R}^N$ such that for all $t \geq 2$ and all A^t, $p(A^t) \leq p(A^{t-1})$. (Note that we reuse the p notation here to represent the pricing in the clock auction: it had earlier referred to pricing

[11] We say that the clock auction is *finite* if there exists some T such the auction always stops by period T. The restriction to finite auctions and finite bid spaces avoids familiar technical difficulties, such as those associated with describing continuous time auctions and with defining dominance solvability for infinite games.

in the threshold auction.)

The clock auction initializes $A_1 = N$. In each period $t \geq 1$, given history A^t, a profile of prices $p(A^t)$ is offered to the bidders. If $p(A^t) = p(A^{t-1})$, the auction stops; bidder i is a winner if and only if $i \in A_t$ and in that case i is paid $p_i(A^t)$. If $i \in A_t$ and $p_i(A^t) < p_i(A^{t-1})$, then i may choose to refuse the new price and exit the set of active bidders.[12] Letting $E_t \subseteq A_t$ denote the set of bidders who choose to exit, the auction continues in period $t+1$ with the new set of active bidders $A_{t+1} = A_t \backslash E_t$ and the new history $A^{t+1} = (A^t, A_{t+1})$.

To complete the description of the auction as an extensive-form mechanism, we also need to describe bidders' information sets. We allow general information disclosure: bidder i observes some signal $\sigma_i(A^t)$ in addition to his current price $p_i(A^t)$ in history A^t.

A strategy for bidder i in a clock auction is a *cutoff strategy* with cutoff b_i if it specifies exit if and only if $p_i(A^t) < b_i$, for some $b_i \leq p_i(N)$. Note in particular that every cutoff strategy accepts the opening price. The next two results show that clock auctions in which bidders are restricted to cutoff strategies (for example, in which they must use proxy bidders with cutoff strategies) are equivalent to deferred-acceptance threshold auctions, meaning that the mapping from the profile of bids or cutoffs to allocations and prices are the same for both auctions.[13]

Proposition 7 *For every deferred-acceptance heuristic with finite bid spaces and threshold pricing, there exists an equivalent finite clock auction in which*

[12] In a variant of the auction, all active bidders $i \in A_t$ may choose whether to exit. Although the results below are the same for the auction in the main text and this variant, the version in the main text is preferred because, when there is a feasibility constraint as in 4, it ensures that the clock auction always yields a feasible outcome.

[13] If we were to implement a deferred-acceptance threshold auction as a multi-round procedure in which some information is disclosed to active bidders between rounds and they are allowed to improve their bids, then the resulting mechanisms would be "survival auctions" like those proposed by Fujishima et al. (1999) for more specific settings. These auctions are strategically equivalent to clock auctions without the restriction to cutoff strategies.

bidders are restricted to cutoff strategies.

Proof. Given bid spaces $B_1, .., B_N$, for each $v \in \mathbb{R}$, let and $v^+ = \min\{b_i \in B_i : b_i > v\}$ and $v^- = \begin{cases} \max\{b_i \in B_i : b_i < v\} & \text{if } v_i > \min B_i, \\ \min B_i - 1 & \text{otherwise.} \end{cases}$. Let the opening prices be $p_i(N) = \max B_i$ for each i. Given a deferred-acceptance heuristic auction with scoring rule s, we construct an equivalent clock auction as follows: The price reduction rule in the clock auction reduces the price to every highest-scoring active bidder by the minimal amount, while leaving prices unchanged for the other bidders:

$$\begin{aligned} p_i\left(A^t\right) &= p_i\left(A^{t-1}\right)^- \text{ if } i \in \arg\max_{j \in A_t} s_j^{A_t}\left(p_j\left(A^{t-1}\right), p_{N \setminus A_t}\left(A^t\right)^+\right) \\ p_i\left(A^t\right) &= p_i\left(A^{t-1}\right) \text{ otherwise.} \end{aligned}$$

Note in particular that the auction maintains $p_i(A^t) = p_i(A^{t-1})$ for all $i \in N \setminus A_t$ – thus memorizing the prices rejected by bidders who have quit, so that their cutoffs can be inferred as $p_i(A^t)^+$.

Then equivalence is easy to see: First, for every history of the clock auction, the next set of bidders to quit in the clock auction is the set of bidders who have the maximum scores among the set of active bidders, so the set of winners is the same in both auctions. Second, if any winning bidder had said "no" to any higher price, it would have exited, so each bidder's final clock price is the highest cutoff it could use to be winning – its threshold price.

Proposition 8 *For every finite clock auction in which bidders are restricted to cutoff strategies, there exists an equivalent deferred-acceptance heuristic with finite bid spaces and threshold prices.*

■

Proof. Given a finite clock auction p, we construct bid spaces and a scoring rule to create an equivalent deferred-acceptance heuristic. We take each bidder i's bid space to be $B_i = \{p_i(h) : h \in H\}$ – the set of possible prices agent i could face in the clock auction (which is a finite set in a finite clock auction).

Next, we construct the scoring rule in the following manner: Holding fixed a set of bidders $S \subseteq N$ and their bids $b_S \subseteq \mathbb{R}^S$, let $A_t(S, b_S)$ denote the set of active bidders in the clock auction at round t in which every bidder $j \in S$ uses cutoff strategy b_j and every bidder from $N \backslash S$ never exits. Formally, initialize $A_1(S, b_S) = N$ and iterate by setting

$$A_{t+1}(S, b_S) = A_t(S, b_S) \setminus \left\{j \in S : b_j > p_j\left(A^t(S, b_S)\right)\right\}.$$

This gives an infinite sequence $\{A_t(S, b_S)\}_{t=1}^{\infty}$, but the sets start repeating at some point (when the clock auction stops).

Now for given A, $b_{N\backslash A}$, $i \in A$, and b_i, define the score of agent i as the inverse of how long he would remain active in clock auction if he uses cutoff b_i and all bidders from $N\backslash A$ use cutoffs $b_{N\backslash A}$, while bidders in $A\backslash \{i\}$ never quit:

$$s_i^A\left(b_i, b_{N\backslash A}\right) = 1/\sup\left\{t \geq 1 : i \in A_t\left(\{i\} \cup (N\backslash A), \left(b_i, b_{N\backslash A}\right)\right)\right\}.$$

(Note that the score is $1/\infty = 0$ in cases in which the auction stops with agent i still active.) This score is by construction nondecreasing in b_i. Also by construction, given a set A of active bidders, the set of bidders to be rejected by the heuristic in the next round ($\arg\max_{i \in A} s_i^A\left(b_i, b_{N\backslash A}\right)$) is the set of bidders who would quit the soonest in the clock auction given that the inactive bidders have used cutoffs $b_{N\backslash A}$. If no more bidders would exit the auction, then all active bidders have the score of zero, so the heuristic stops. Finally, as argued above, the winners' clock auction prices are their threshold prices: the winner would have lost by using any higher cutoff in B_i than its clock auction price. ■

4 (Group) Strategy-proofness

Definition 9 *In a clock auction, agent i with value v_i is said to 'bid truthfully" if he accepts clock price if and only if $p_i(h) > v_i$. (Equivalently, if the agent uses a cutoff strategy with cutoff $v_i^+ = \min\{p_i(h) : h \in H,\ p_i(h) > v\}$.)*

Definition 10 *An auction is "weakly group strategy-proof" if for every profile of values v and every set of players $S \subseteq N$ and every strategy profile σ_S of these players, at least one bidder in S has a weakly higher payoff from the profile of truthful bids v_N than from the strategy profile $(v^+_{N\setminus S}, \sigma_S)$.*

Remark 11 *Clock and threshold auctions are not generally "strongly" group strategy-proof, because a bid increase by a losing bidder that increases a winner's threshold price is strictly profitable for the winner and weakly profitable for the loser.*

Clock auctions can have various information disclosure policies, leading to a potentially large set of strategies for bidders, but always including the cutoff strategies. The definition of group strategy-proofness applies to all such auctions.

Proposition 12 *Every finite clock auction (with any information disclosure) is weakly group strategy-proof.*

Proof. Consider the first stage of clock auction affected by a group deviation. If at that stage, the deviation is by a bidder who chooses to exit, then his deviation payoff is zero, so he does not benefit from the group deviation. The other possibility is that the deviation is by a bidder who chooses not to exit at a price equal to or below his truthful value, but such a bidder either eventually exits or wins and receives his final clock price, which in a descending clock auction cannot be higher and so cannot exceed his value. Hence, this deviator's payoff is non-positive. In both cases, at least one participant in the group deviation fails to gain from the deviation. ■

The preceding argument is independent of the information policy in the auction, so it works for larger sets of strategies than just the cutoff strategies. In particular, the clock auction is still group strategy-proof when bidders are restricted to play cutoff strategies. Combining the two previous propositions, we get

Corollary 13 *Any deferred-acceptance auction with threshold prices and finite bid spaces is weakly group strategy-proof.*

It is possible to establish the result directly, without using the equivalence to clock auctions or restricting attention to finite bid spaces.

5 Substitutability -> Heuristic implementation

We say that a clock auction *implements allocation rule* $\alpha : B \to 2^N$ if (i) $\{p_i(h) : h \in H\} = B_i$ for each i and (ii) the auction generates $\alpha(b)$ when bidders use cutoff strategies with cutoffs $b \in B$. Also, the allocation rule α is *monotonic* if $i \in \alpha(b)$ and $b_i' < b_i$ implies $i \in \alpha(b_i', b_{-i})$. It *has substitutes* if $i \in \alpha(b)$ and $b_j' > b_j$ for some $j \neq i$ implies $i \in \alpha\left(b_j, b_j'\right)$.

A set $S \subseteq 2^N$ of subsets of N has *no disposal* if for all $A, A' \in S$, $A \subseteq A'$ implies $A = A'$.

Proposition 14 *With finite bid spaces, any monotonic allocation rule α with substitutes whose range $\alpha(B)$ has no disposal can be implemented with a clock auction or a deferred-acceptance heuristic.*

Proof. α can be implemented with a clock auction described as follows: For each i, set $p_i(N) = \max B_i$ and then in each period t, set

$$\begin{aligned} p_i\left(A^t\right) &= p_i\left(A^{t-1}\right)^- \text{ if } i \in A_t \backslash \alpha\left(p_{A_t}\left(A^{t-1}\right), p_{N\backslash A_t}\left(A^{t-1}\right)^+\right) \\ p_i\left(A^t\right) &= p_i\left(A^{t-1}\right) \text{ otherwise.} \end{aligned}$$

(That is, decrement prices to those bidders who wouldn't win given the current best offers – the current prices for the active bidders, and the last prices accepted by the bidders who have exited.)

To see that this auction implements α, observe that if bidders use cutoff strategies with cutoffs $b_i \in B_i$, then a bidder $i \in \alpha(b)$ can never exit the auction: when $i \in A_t$ and he is offered price $p_i(A^t) = b_i$, we will have $p_{A_t \setminus \{i\}}(A^{t-1}) \geq b_{A_t \setminus \{i\}}$ and $p_{N \setminus A_t}(A^{t-1})^+ = b_{N \setminus A_t}$, hence the substitute property and $i \in \alpha(b)$ imply $i \in \alpha\left(p_{A_t}(A^{t-1}), p_{N \setminus A_t}(A^{t-1})^+\right)$ and so his price is not decremented. Thus, we have $\alpha(b) \subseteq A_t$ throughout the auction. On the other hand, when the auction stops we have $A_t \subseteq \alpha\left(p_{A_t}(A^{t-1}), p_{N \setminus A_t}(A^{t-1})^+\right)$, and putting together with the previous inclusion and using the no-disposal of the range of α implies $\alpha(b) = A_t = \alpha\left(p_{A_t}(A^{t-1}), p_{N \setminus A_t}(A^{t-1})^+\right)$, hence the auction implements α. ■

The assumption of substitutes is not dispensable in the above proposition: in Example 22 below, we will see a monotonic allocation rule whose range has no disposal that cannot be implemented by a clock auction. While many deferred-acceptance heuristic allocation rules do satisfy substitutes, not all of them do. For example, consider the allocation rule $\alpha(b_1, b_2) = \begin{cases} \{1,2\} & \text{if } b_1 < 1, \\ \varnothing & \text{otherwise.} \end{cases}$ This allocation does not have substitutes, but is implementable with the deferred-acceptance heuristic with the scoring rule $s_1^{\{1,2\}}(b_1) = \max\{b_1 - 1, 0\}$, $s_2^{\{2\}}(b_2, b_1) = 1$, and $s_2^{\{1,2\}}(b_2) = s_1^{\{1\}}(b_2, b_1) = 0$.)

The no-disposal assumption is also indispensable, which is illustrated by the allocation rule $\alpha(b) = \arg\min_{i \in N} b_i$, with $B_1 = ... = B_N$ (so that ties exist, and in case of ties all the tied bidders win). Then there is no clock auction implementing α.[14] The no-disposal assumption is satisfied, e.g., in

[14] Indeed, any such auction would start with equal prices, and it would then not be "safe" to reduce any price: if all bidders have set their cutoff equal to the common price, then the reduction would eliminate any affected bidder. On the other hand, if some one bidder has bid below the common price, then failing to reduce his price prevents the algorithm from ever identifying that bidder.

Example 4 if the feasible set F is *comprehensive* (meaning that $A \in F$ implies $A' \in F$ whenever $A \subseteq A'$) and the heuristic has perfect feasibility checking. But not all heuristic allocation rules satisfy it (e.g., in Example 5, the heuristic may accept a set A' when bids are low and a set $A \subset A'$ when bids are high).

6 Pay-as-Bid: Full-info equivalence

Recall that for any finite bid space B and allocation rule α, the threshold prices for winners are given by $p_i(b_{-i}) = \max\{b_i' \in B_i : i \in \alpha(b_i', b_{-i})\}$. In particular, $i \in \alpha(p_i(b_{-i}), b_{-i})$.

Proposition 15 *Every paid-as-bid deferred-acceptance auction with finite bid sets B_i for all values $v_i < \max B_i$ has a complete-information Nash equilibrium profile in which, for each $i \in N$, the bids are $b_i = \max\left\{v_i^+, p_i\left(v_{-i}^+\right)\right\}$ and in which the resulting allocation is $\alpha(b) = \alpha(v^+)$.*

Proof. Since changing accepted bids so that they are still accepted does not affect the deferred-acceptance heuristic's outcome, we have $A \equiv \alpha(v^+) = \alpha\left(p_A(v^+), v_{N\backslash A}^+\right) = \alpha(b)$ and $p_i(b_{-i}) = p_i\left(p_{A\backslash i}(v^+), v_{N\backslash A}^+\right) = p_i\left(v_{-i}^+\right) \geq v_i^+$ for each $i \in A$. Now, we verify that the bids constitute a Nash equilibrium. Every bidder $i \in A$ is winning and receiving payment of $b_i = p_i\left(v_{-i}^+\right) = p_i(b_{-i}) \geq v_i^+$, and any larger bid by i would be losing, so a winning bidder i has no profitable deviation. Every bidder $i \in N\backslash A$ is losing with its bid of v_i^+, and so any winning bid for i earns a negative payoff: a losing bidder has no profitable deviation. ■

Next, we introduce a pair of standard definitions.

Definition 16 *An auction is* dominance-solvable *in state v if under full information, there exists a payoff profile that is the unique outcome of iterated deletion of (weakly) dominated strategies, regardless of the order of elimination.*

For "generic" values ($v_i \notin B_i$ for each i), a unique payoff profile implies a unique outcome (allocation and winning bids).

Definition 17 *Assignment rule* α *is* non-bossy *if for any* $i \in N$, $b \in B$ *and* $b'_i \in B_i$, $\alpha(b'_i, b_{-i}) \cap \{i\} = \alpha(b) \cap \{i\}$ *implies* $\alpha(b'_i, b_{-i}) = \alpha(b)$.

Non-bossiness means simply that a bidder cannot affect others' allocations without changing his own allocation. Some deferred-acceptance heuristics are non-bossy, but as our "reference pricing" example illustrates, some are not. In a deferred-acceptance-heuristic, a winner who changes its bid without changing its winning status (that is, agents $i \in \alpha(b_i, b_i) \cap \alpha(b'_i, b_i)$) can never affect others' winning status, but because bidder's scores can depend on losing bids, a loser who changes to a different losing bid ($i \notin \alpha(b_i, b_i) \cup \alpha(b'_i, b_i)$) may affect the set of winners.

Proposition 18 *Consider a paid-as-bid auction with a monotonic, non-bossy assignment rule* α *and finite bid spaces* B. *Say that a value profile* v *is "generic" if for each* i, $v_i \in [0, \bar{v}_i] \backslash B_i$.

(i) The auction is pure-strategy dominance-solvable for all generic value profiles if and only if α *can be implemented via a deferred-acceptance heuristic.*

(ii) In this case, for every generic value profile, the unique payoff profile surviving iterated deletion of dominated strategies is also the unique (pure or mixed) Nash equilibrium payoff profile in undominated strategies.

(iii) In this case, one strategy profile that survives iterated deletion of dominated strategies and is a Nash equilibrium in undominated strategies is the one described in Proposition 15.

Remark 19 *We need* α *to be non-bossy to guarantee a unique outcome because of examples like the following one, which arises as a particular instance of yardstick competition. Let* $N = 2$, $B_1 = \{1, 3\}$, $B_2 = \{2, 4\}$, $\alpha(b) = \{1\}$ *if* $b_2 = 4$, *and* $\alpha(b) = \varnothing$ *otherwise. With* $v_1 < 3$, *bidder 1's dominant strategy is* *to bid* 3*, but bidder 2 has no dominated strategies. The two strategy profiles that survive iterated elimination in this example are also the undominated Nash equilibrium profiles: they are* $(3,4)$ *(in which bidder 1 wins) and* $(3,2)$ *(in which there is no winner). In this example and many others, iterated dominance and undominated Nash equilibrium fails to nail down the losing bids in a paid-as-bid auction game, leading to a multiplicity of possible outcomes. When* α *is non-bossy, the multiplicity of possible losing bids is irrelevant, because one can deduce the unique auction outcome just by reasoning about winning bids, as shown in the proof below.*

Proof. For the "if" direction of (i), recall from Proposition 7 that any assignment rule α that is implementable via a deferred-acceptance heuristic is also implementable with a clock auction in which bidders use cutoff strategies with cutoffs corresponding to their bids in the deferred-acceptance heuristic. Furthermore, we can implement assignment rule α *with paid-as-bid pricing* using the following "two-phase clock auction": In phase 1, the the clock auction described above is run to determine the set of winners. In phase 2, the payments to the winners are determined by allowing prices to continue falling (through points in B_i) until all bidders "quit", with the winners being paid the last prices they accept. The two-phase clock auction game in which bidders use the cutoff profile b obviously leads to the same outcome as the paid-as-bid sealed-bid auction game based on the deferred-acceptance algorithm in which the bid profile is b.

If the assignment rule is non-bossy, then for generic values $v_i \in \mathbb{R}\backslash B_i$ the game satisfies the TDI condition of Marx and Swinkels (1997), and so the payoffs profiles surviving iterated deletion do not depend on the order of deletion: hence deleting dominated or equivalent strategies in any order leads to the same set of possible outcomes.[15] We specify the following deletion process:

[15] We say that given strategy sets $\hat{B}_i \subseteq B_i$ for each i two strategies $b_i, b_i' \in \hat{B}_i$ of agent i are equivalent if $\alpha(b_i, b_{-i}) = \alpha(b_i', b_{-i})$ for all $b_{-i} \in \hat{B}_{-i}$. Obviously deletng strategies that are equivalent to surviving ones does not affect the solution to iterated deletion of

Begin by deleting for each agent i all the bids/cutoffs $b_i < v_i^+$ (which are either dominated by or equivalent to the bid v_i^+). In the game that remains after these initial deletions, every bidder strictly prefers any outcome in which it wins to any in which it loses. We specify the next deletions inductively by referring to the sequence of prices $\{p(A^t)\}$ that would emerge during phase 1 if each bidder were to use the cutoff strategy v_i^+. At the beginning of each step t of our iterated deletion process, the set of strategies remaining to each bidder i is $\hat{B}_i^{t-1} = B_i \cap [v_i^+, \max\left\{v_i^+, p_i\left(A^{t-1}\right)\right\}]$. As the prices are reduced to $p\left(A^t\right)$, for each bidder i all the cutoffs $b_i \in \hat{B}_i^{t-1}$ such that $b_i > \max\left\{v_i^+, p_i\left(A^t\right)\right\}$ are sure to lose and are therefore either dominated by or equivalent to the cutoff v_i^+, hence we can let $\hat{B}_i^t = B_i \cap [v_i^+, \max\left\{v_i^+, p_i\left(A^t\right)\right\}]$. The iterations continue until phase 1 ends and the winners are determined at the end of some iteration T to be $\alpha(\max \hat{B}^T)$. For each agent i, if $\hat{B}_i^T$ is not a singleton, then its largest element, $\max \hat{B}_i^T = \max(v_i^+, p_i\left(A^T\right))$, is dominant in the game with just the bids $\hat{B}^T$ (because it wins at the highest price). So, we may do one more round of deletions, taking $\hat{B}_i^{T+1} = \{\max(v_i^+, p_i\left(A^T\right))\}$. Hence, the single outcome of iterative elimination of undominated cutoffs is the one for the bid profile $(\max(v_i^+, p_i\left(A^T\right)))_{i \in N}$.

For (ii), fix an undominated mixed Nash equilibrium profile**.** For each bidder i with a zero equilibrium payoff, all bids of v_i^+ or more must be always losing. Hence, by non-bossiness, we may replace every such bidder i's bids by the pure strategy bid v_i^+ to obtain another mixed strategy profile σ with the same distribution of outcomes. We show below that σ is actually a pure strategy bid profile, and specifically it is the profile $(\max(v_i^+, p_i\left(A^T\right)))_{i \in N}$ that results from iterated elimination of weakly dominated strategies, as described above.

For any bidder i with strictly positive equilibrium expected payoffs, all bids in the support of σ_i have positive expected payoffs, so all must win

dominated strategies. Note furthermore that given non-bossiness, such equivalence obtains whenever agent i's own allocation does not change, i.e. $\alpha\left(b_i, b_{-i}\right) \cap i = \alpha\left(b_i', b_{-i}\right) \cap i$ for all $b_{-i} \in \hat{B}_{-i}$.

with a positive probability against σ_{-i}. Consider the maximum bid profile in the support of σ. Referring to the clock auction process, we infer that if any positive-payoff bidder's bid is losing for that profile, then it is losing for all profiles in the support of σ, which contradicts positive expected payoffs. Since reducing a winning cutoff/bid in the clock auction does not affect the allocation, for every bid profile in the support of σ, the positive-payoff players are the winners. Since the highest always-winning bid earns strictly more than any lower winning bid, this further implies that the winners' equilibrium mixtures are degenerate: winning bidders play pure strategies. Therefore, σ assigns probability one to some single bid profile b.

Next, we claim that the iterative deletions described in the proof of (i) above do not delete any of the component bids in b. Phase I of the iterative deletion procedure deletes only bids above v_i^+ for zero-payoff bidders and only always-losing bids for positive-payoff bidders, so all the component bids in b survive that phase. Phase II deletes all but the highest remaining bid of each winning bidder: the lower bids are never best replies to the highest surviving bids (they always win, but they are paid less). Hence, the full procedure never deletes any component bid in the profile b. It follows that $b = (\max(v_i^+, p_i\left(A^T\right)))_{i \in N}$ and that the outcome of b is the outcome of every undominated Nash equilibrium.

To prove (iii): in the surviving bid profile b, each agent $i \in A^T$ bids its threshold price, which is $p_i\left(A^T\right) \leq v_i^+$, while each $i \in N \backslash A^T$ bids v_i^+, which is by definition above its threshold price. Thus by Proposition 15 it is a Nash equilibrium and it contains only undominated strategies, and as argued above it survives iterated deletion of dominated strategies.

It remains to prove the "only if" direction of (i): we assume that the auction is dominance solvable and relate a sequence of sets $\hat{B}\left(A^t\right)$ surviving a number of rounds of iterated elimination of dominated strategies to a corresponding sequence of clock prices $p\left(A^t\right)$ that implements α. Importantly, our construction has the properties for each i and any legal history A^t of the auction that (a) $\max \hat{B}_i(A^t) = p_i(A^{t-1})$ for every $i \in A_t$, $\min \hat{B}_i(A^t) > p_i(A^{t-1})$ for every $i \in N \backslash A_t$, and $\alpha(b) \subseteq A_t$ for all $b \in \hat{B}(A^t)$, and (b) the strategy sets $\hat{B}(A^t)$ are determined by iterated deletion of dominated and equivalent strategies in a particular order for *any* generic value profile v such that $[v_i^+ \leq p_i(A^{t-1})$ if and only if $i \in A_t]$. We establish properties (a) and (b) by induction.

We initialize the construction with clock prices $p(N) = \max B$ and sets of profiles $\hat{B}(N) = B$. For each clock round $t = 1, 2, 3, ...$, given any legal history A^t and previously determined strategy profiles $\hat{B}(A^t)$, we build $p(A^t)$ and $\hat{B}(A^{t+1})$ as follows. Within each "clock" iteration t, we nest a second iteration employing a dummy variable $\bar{B}$. Initialize $\bar{B} = \hat{B}(A^t)$. Check whether there is some $i \in A_t$ and $b_i, b_i' \in \bar{B}_i$ such that $b_i' > b_i$ and $\alpha(b_i', b_{-i}) = \alpha(b_i, b_{-i})$ for all $b_{-i} \in \bar{B}_{-i}$. If there is, we delete b_i from $\bar{B}_i$. Notice that the bid b_i is dominated by or equivalent to b_i' for all value profiles v (it wins against the same profiles b_{-i} and earns a higher price when it wins), so this step deletes only equivalent or weakly dominated strategies. Repeat this step to further trim $\bar{B}$ until the checking step indicates that no such qualifying bids $b_i' > b_i$ remain.

We claim that, after maximal trimming of $\bar{B}$, either all remaining strategy profiles lead to the same winners (i.e. $\alpha(\bar{B}) = \{A_t\}$) or else there exists an agent $i \in A_t$ for whom the bid $p_i(A^{t-1}) = \max \bar{B}_i$ always loses, that is, $i \in N \backslash \alpha(p_i(A^{t-1}), b_{-i})$ for all $b_{-i} \in \bar{B}_{-i}$. To establish this claim, we use the inductive property and the assumption of dominance solvability for the game with a value profile v satisfying $v_i^+ < \min \bar{B}_i$ for $i \in A_t$ (so that agents $i \in A_t$ always strictly prefer to win) and $v_i^+ > p_i(A^{t-1})$ for $i \in N \backslash A_t$ (the remaining agents prefer to lose). First, by inductive property (b), iterated deletion of dominated and equivalent strategies for v yields the sets $\hat{B}(A^t)$. Next, given such sets, if there are any two bids $b_i, b_i' \in \hat{B}_i(A^{t-1})$ such that $b_i' > b_i$ and b_i is dominated by b_i', then (by monotonicity) both win against the same set of opposing bid profiles b_{-i} and hence (by non-bossiness) lead to the same allocations. Bids b_i that are dominated in this way are eliminated by the iterative "pruning" described above, at the end of which none such remain in $\bar{B}$. Hence, unless there is a unique set of winners ($\alpha\left(\bar{B}\right) = \{A_t\}$), dominance solvability for value profile v implies that there is another dominance relation to be found: there exists at least one active bidder $i \in A_t$ and bids $b_i, b_i' \in \bar{B}_i$ with with $b_i' < b_i$ such that b_i is dominated by b_i'. Given v, such dominance is possible only if b_i never wins, which by monotonicity implies our claim that $p_i\left(A^{t-1}\right) = \max \bar{B}_i \geq b_i$ never wins.

For iteration t of the clock auction, we reduce the price to the identified bidder i by letting $p_i\left(A^t\right) = \max\left(\bar{B}_i \backslash \left\{p_i\left(A^{t-1}\right)\right\}\right)$ and $p_j\left(A^t\right) = p_j\left(A^{t-1}\right)$ for every bidder $j \in N \backslash \{i\}$. The clock auction and the strategy sets for the next round are then updated as follows. If bidder i accepts the reduced clock price $p_i\left(A^t\right)$ at iteration t, we let $A_{t+1} = A_t$ and $\hat{B}_i\left(\left(A^t, A_t\right)\right) = \bar{B}_i \backslash \left\{p_i\left(A^{t-1}\right)\right\}$. If, instead, bidder i quits, we let $A_{t+1} = A_t \backslash \{i\}$ and $\hat{B}_i\left(\left(A^t, A_t \backslash \{i\}\right)\right) = \left\{p_i\left(A^{t-1}\right)\right\}$. For all bidders $j \in N \backslash \{i\}$, regardless of i's decision to accept or reject, we let $p_j\left(A^t\right) = p_j\left(A^{t-1}\right)$ and $\hat{B}_j\left(\left(A^t, A_t\right)\right) = \hat{B}_j\left(\left(A^t, A_t \backslash \{i\}\right)\right) = \bar{B}_j$. This guarantees that property (b) extends to both history $\left(A^t, A_t\right)$ and $\left(A^t, A_t \backslash \{i\}\right)$.

To see that with this construction, property (a) also extends from t to $t+1$, observe that it suffices to check the property for the bidder i whose price is changed. If $v_i^+ \leq p_i\left(A^{t-1}\right)$, then the bidder remains active and $\max \hat{B}_i\left(A^{t+1}\right) = p_i\left(A^t\right)$, as specified by the inductive property. Otherwise, $v_i^+ > p_i\left(A^{t-1}\right)$, the bidder quits and $\hat{B}_i\left(A^{t+1}\right) = \left\{p_i\left(A^t\right)\right\}$, so $\min \hat{B}_i\left(A^{t+1}\right) > p_i\left(A^t\right)$. By (a), the clock auction with cutoffs b leads to the outcome $\alpha(b)$. ■

Here are two examples of non-bossy allocation rules:

Example 20 (Optimization) *Letting $F \subseteq 2^N$ be the feasible set as in Example 4, the optimizing allocation rule is given by*

$$\alpha\left(b\right) \in \arg\min_{A \in F} \textstyle\sum_{i \in A} b_i.$$

It is easy to see that, if B rules out ties (so $\arg\min$ *is always single-valued), optimizing allocation rules are non-bossy, because:*

$$\text{For } i \notin \alpha(b_i, b_i) \cup \alpha(b_i', b_i) \text{ we have } \alpha(b_i, b_i) = \arg \min_{A \in F : i \notin A} \textstyle\sum_{j \in A} b_j = \alpha(b_i', b_i)$$
$$\text{For } i \in \alpha(b_i, b_i) \cap \alpha(b_i', b_i) \text{ we have } \alpha(b_i, b_i) = \arg \min_{A \in F : i \in A} \textstyle\sum_{j \in A \setminus \{i\}} b_j = \alpha(b_i', b_i)$$

Example 21 (Fixed Scoring and Perfect Feasibility Checking) *Suppose we are in the setting of Example 4, that the feasible set $F \subseteq 2^N$ is comprehensive (as defined above), and that* $s_i^A\left(b_i, b_{N\setminus A}\right) = \begin{cases} \sigma_i(b_i) & \text{if } A \cup \{i\} \in F, \\ 0 & \text{otherwise,} \end{cases}$ *where the functions $\sigma_i(b_i)$ are increasing and positive-valued and there are no ties (so feasibility is always maintained). As observed above, every deferred-acceptance procedure satisfies non-bossiness for the winning bids. To check that condition for rejected bids, too, suppose that given bid profile b agent i's bid b_i is rejected in round t and agent j's bid b_j is rejected in round t (hence $A_t \setminus \{i, j\} \in F$, and so by comprehensiveness $A_t \setminus \{j\} \in F$) but replace b_i with a bid $b_i' < b_i$ that is rejected in round $t+1$. In this case, bid j must be rejected in round t (so we must have*

$$\max_{k \in A_t \setminus \{i,j\} : A_t \setminus \{j,k\} \in F} \sigma_k(b_k) < \sigma_i(b_i') < \sigma_j(b_j).$$

After round $t+1$ the heuristic is unaffected by the replacement. Iterating this argument, we see that any change in b_i that preserves this bid being losing will not affect the allocation produced by the heuristic.

7 Comparisons to Properties of Other Auctions

The properties that we have derived for deferred-acceptance auctions are not shared by other classes of auctions that have received close attention. Below are some examples to show that our findings do not apply to auctions in which winners are selected using either optimization or a greedy-acceptance heuristic.

7.1 Auctions Using Optimization

An optimizing allocation rule minimizes the total social cost subject to a feasibility constraint. Letting $F \subseteq 2^N$ be the feasible set as in Example 4, an optimizing rule solves

$$\alpha(b) \in \arg\min_{A \in F} \textstyle\sum_{i \in A} b_i.$$

This is a monotonic allocation rule, and if $B_i = (0, +\infty)$ then the threshold prices are Vickrey prices - the agent is paid the externality his inclusion creates on the other agents:

$$p_i(b_{-i}) = \min_{A \in F : A \subseteq N \setminus \{i\}} \textstyle\sum_{j \in A} b_j - \sum_{j \in \alpha(b) \setminus \{i\}} b_j$$

(so that his surplus $p_i(b_{-i}) - b_i$ captures the entire social cost savings due to his participation).

In some circumstances, the optimizing allocation rule and Vickrey prices can be computed with a deferred-acceptance heuristic or clock auction (ignoring any computational challenges that this might involve). This is determined by properties of the feasible set F. For example, when F is a comprehensive set and $\min B_i > 0$ for each i, the range of α has "no disposal" (as defined above). If α also satisfies substitutes, then by Proposition 14 allocation rule α is implementable by a clock auction or a deferred-acceptance heuristic when bid spaces B_i are finite. (See Bikhchandani et al. (2011) for conditions on F for an optimizing allocation rule to satisfy substitutes; see also Ausubel (2004) and de Vries and Vohra (2007) for earlier examples of settings in which optimizing allocation rules can be implemented via clock auctions.) In this case, paid-as-bid equivalence also holds. Bernheim and Whinston

(1986) had shown payoff equivalence between Vickrey and paid-as-bid auctions when bidders are substitutes using a coalition-proofness refinement to select among Nash equilibrium. Our analysis finds the same conclusion under different assumptions and conditions. We use either iterated dominance or undominated Nash equilibrium to select a Nash equilibrium and we allow a wide range of heuristic allocation rules with substitution, but we limit attention to environments with single-minded bidders.

For an example in which the substitutes condition does not hold and so an optimizing α cannot be implemented via a deferred-acceptance heuristic, consider the following:

Example 22 $N = \{1,2,3\}$ *and* $F = \{\{1,2\},\{3\}\}$*. Intuitively, the structure of* F *makes bidders* 1 *and* 2 *complementary. In this case,* $\alpha(b) = \{1,2\}$ *if* $b_1 + b_2 < b_3$ *and* $\alpha(b) = \{3\}$ *if* $b_1 + b_2 > b_3$*. In any deferred-acceptance heuristic, the first bid to be rejected can be based only on pairwise comparisons of bids, so it cannot be generally consistent with the preceding inequalities.*

Observe, too, that the Vickrey auction implementing this allocation rule does not satisfy either weak group strategy-proofness or paid-as-bid equivalence. For example, when $b_1 + b_2 < b_3$, the Vickrey prices are $p_1(b_2, b_3) = b_3 - b_2$ and $p_2(b_2, b_3) = b_3 - b_1$. Then the two winners have a strictly improving coalitional deviation in which they bid $b_1 < v_1$, $b_2 < v_2$ such that $b_1 + b_2 < v_3$: both still win but each is paid strictly more. Also, note that the sum of Vickrey prices is $2b_3 - b_1 - b_2 > b_3$, but in the corresponding paid-as-bid auction, there cannot be a Nash equilibrium in which bidders 1 and 2 win and are paid a total of $b_1 + b_2 > v_3$, since then bidder 3 would deviate to undercut them. (In fact, in all the Nash equilibria in which bidder 3 uses undominated strategies and bidders 1 and 2 win, they together pay $b_1 + b_2 = v_3$. These outcomes have been identified by Bernheim and Whinston (1986).) So, the Vickrey mechanism appears "too expensive" in this case relative to optimization with paid-as-bid pricing. One solution that has been proposed to the problem of Vickrey auctions' excessive costs (insufficient revenues) is "core-selecting auctions" (Day and Milgrom 2008), which sacrifice strategy-proofness even for single-minded bidders. Deferred-acceptance heuristics offer a possible alternative way to reduce costs (increase revenues), which preserves strategy-proofness.

7.2 Auctions Using Greedy-Acceptance Heuristics

To compare to the greedy-acceptance heuristics auctions of LOS, consider again Example 22. For illustration, let a bidder's score be its bid, so the heuristic iterates accepting the highest bid that is still feasible. If we break ties in favor of lower-numbered agents, we have $\alpha(b) = \{b_1, b_2\}$ if $\min\{b_1, b_2\} \geq b_3$, and $\alpha(b) = \{b_3\}$ otherwise. Suppose bid spaces are $b_i = [0, \bar{b}]$. The threshold payments for the reverse auction are as follows: First, if $\alpha(b) = \{b_1, b_2\}$, then for $i = 1, 2$, $p_i = \bar{b}$ if $b_{-i} \leq b_3$ and $p_i = b_3$ otherwise. Second, if $\alpha(b) = \{b_3\}$, then $p_3 = \min\{b_1, b_2\}$.

Observe that this allocation rule cannot be implemented with a deferred-acceptance heuristic or a descending clock auction, since the allocation is completely determined by the single "best" (lowest) bid while the first step of the deferred-acceptance heuristic is determined by the single worst bid according to some criterion. One might conjecture that greedy-acceptance heuristics could instead be matched with an ascending clock auction, but that fails, too, because when bidder 3 exits first, the allocation is determined to be $\alpha(b) = \{b_1, b_2\}$ but the prices to the winners are not yet determined.[16]

Next, observe that the greedy-acceptance threshold auction fails weak group strategy-proofness. For example, if $v_1, v_2 > v_3 > 0$, bidders 1 and 2

[16] A traditional purpose of a clock auction is to economize on information transmission or conceal some information, and accordingly we require that that clock-auction prices stop changing once the allocation has been determined. Without this condition, any allocation rule can be implemented with a clock auction, simply by running all the prices down to elicit complete information about cutoffs from all bidders and determining the allocation as a function of those.

could jointly deviate to bid $b_1, b_2 < v_3$, which will give each of them threshold prices of $\bar{b}$.

Finally, the threshold and paid-as-bid auctions based on the greedy-acceptance heuristic do not have the outcome equivalence properties described above. For suppose that $v_1, v_2 < v_3$. In the threshold auction, bidders 1 and 2 win and their threshold payments are both $\bar{b}$, but a paid-as-bid auction with complete information cannot have a pure Nash equilibrium in which bidders 1 and 2 win and both get paid above v_3, since then bidder 3 would deviate to undercut them both and win. So a greedy-acceptance heuristic with threshold payments is more expensive in this case than any pure Nash equilibrium of its paid-as-bid counterpart.

8 Discussion

This paper describes parts of the analysis that were used to develop one of the options for a "reverse auction" to purchase TV broadcast licenses as part of the FCC's incentive auction program. The full incentive auction program also included two other important parts: the "forward auction," in which rights for high-speed wireless broadband would be sold, and the "clearing rule" in which the results of the forward and reverse auctions are combined to determine the quantities to be transacted. Neither of those pieces is described here. Also omitted is a description of the development by others of algorithms and procedures to make the very fast computations required by this particular auction.

Several of the game-theoretical and economic analyses of auctions developed above contribute in significant ways to the practical evaluation of some proposed incentive-auction designs.

1. The ability to implement the heuristic threshold auction with a clock auction is important in practice for two reasons. The first is familiarity: most past FCC auctions have used the simultaneous multiple round

auction design, which closely resembles a clock auction. The second is that, for single-minded bidders, the dominant strategy property is much more obvious for clock auctions than for sealed-bid threshold auctions.[17] Also, sealed-bid threshold auctions are strategy-proof only if bidders trust the auctioneer to compute the threshold prices correctly, while clock auctions are strategy-proof and (weakly) group strategy-proof regardless of the clock adjustment rule.

2. Strategy-proofness is important because it reduces the bidders' costs of participating, especially for small local broadcasters whose participation is needed for a successful incentive auction. Also, it eliminates the losses from mistakes that can occur in other kinds of auctions such as paid-as-bid auctions, in which a bidder's optimal bid necessarily depends on a guess about how others will bid.

3. Outcome equivalence is the only available indicator of the extra cost of providing dominant-strategy incentives, compared to a paid-as-bid rule. The estimated cost that it implies for auctions based on deferred-acceptance heuristics is *zero*.

4. The ability to incorporate a cost target into the price determination rule may be important for the incentive auction, because of the way the reverse- and forward-auction outcomes are combined. Markets clear only when the total cost of procuring licenses sufficient to clear all broadcasters from a set of channels is sufficiently lower than the corresponding forward auction revenue.[18]

5. Yardstick competition is extremely important in practice. It allows the

[17] Kagel (1987) makes a similar point, providing experimental evidence that bidders in English auctions may make fewer errors than in the equivalent sealed-bid second-price auctions.

[18] By the law authorizing the auctions, the sale can proceed only if the net revenues are sufficient to pay the relocation costs of broadcasters who do not sell, plus the cost of the FirstNet public safety system, plus a target to be specified by the regulator.

FCC to set reserve prices based on market information in regions where there are too many constraints for direct competition to discipline auction prices.

6. Finally, the fact that deferred-acceptance heuristics can be used to implement any bid-selection rule with the substitutes and no-disposal properties is useful, because the most costly constraints in the problem are the ones limiting on the number of available channels in each metropolitan area, which enforces a pattern of substitution among broadcasters.

This formal results in this paper are derived only for the case of single-minded bidders. In the actual FCC auction problem, the single-mindedness property describes individual station owners who decide only between selling their broadcast licenses or continuing on-air. The actual auction, however, is likely to include additional options, such as allowing a broadcaster to switch to broadcasting to the same population but in a less-congested, lower-frequency band. Also, some broadcasters own multiple stations and may wish to contemplate which subset of stations to sell. These are just two examples of *multi-minded* bidders and the actual auction with multi-minded bidders would need to deviate from the theoretical version described here.

Clock auctions have been used and studied for multi-minded bidders, but for those bidders truthful bidding is often not optimal. Milgrom (2000) and Gul and Stacchetti (2000) examine simple heuristic clock auctions (for example, forward auctions that raise prices for overdemanded goods) under the assumption that bidders bid "straightforwardly," showing that with such behavior, these auctions achieve efficiency when goods are substitutes. But straightforward bidding is not generally consistent with equilibrium for multi-minded bidders in this design (Weber, 1997). Some clock auctions do provide incentives for truthful bidding even for multi-minded bidders by replicating the Vickrey outcome (see Ausubel (2004) and Bikhchandani et al (2011)), but those can entail a significant computational burden. Bartal et al. (2003) propose a clock auction that trades off efficiency for a reduced computational burden. In that proposal, each bidder is asked for its demand only once, and the prices quoted to a bidder depend on what the previous bidders have chosen. Under some conditions, such an auction may achieve "approximate" efficiency.

In classical partial equilibrium theory, markets bring together buyers and sellers of a single homogeneous good, all of whom know the relevant prices of all other goods and have declining marginal values of additional quantity. In the transactions we have analyzed, this description is very far off the mark. In the incentive auction transaction and some others, large numbers of complex constraints or complicated valuations create new challenges for economists engaged in market design. This paper reports economic theory that has been helpful to address one such challenge, in the hopes that it may be directly or indirectly useful for other applications as well.

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